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On Partitioning Colored Points

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When Can We Divide Colored Points along the Colors by Hyperplanes?

Let X be a finite subset of $\mathbb{R}^d$, and suppose that each point in X is painted with one of k colors. We say that a subset S of X can be *partitioned along the colors by hyperplanes* if there is a collection $\mathcal{F}$ of hyperplanes satisfying the following conditions:

- every hyperplane in $\mathcal{F}$ avoids the points in S;
- every two points in S with different colors can be separated by some hyperplane in $\mathcal{F}$;
- no hyperplane in $\mathcal{F}$ separates points in S with the same color.

In particular, for the case of 2 colors, we say that S can be *separated along the colors*.

Kirchberger's Theorem [2]

Let X be a finite subset of $\mathbb{R}^d$, and suppose that each point in X is painted with one of 2 colors. If every $d+2$ or fewer points in X can be separated along the colors, then all the points in X can be separated along the colors.

Basic Notions and Results

For a finite subset X of $\mathbb{R}^d$, we denote by $\mathcal{H}(X)$ the set of all partitions of X that can be realized[1] by hyperplanes.

We introduce key notions.

Definition

- A subset $\mathcal{S}$ of $\mathcal{H}(X)$ is a *full subdivision* of $\mathcal{H}(X)$ if every distinct two elements in X can be separated by some member of $\mathcal{S}$.
- A subset of $\mathcal{H}(X)$ is called a *transversal* for the full subdivisions of $\mathcal{H}(X)$ if it intersects all the full subdivisions of $\mathcal{H}(X)$.

Notation

We denote by $\tau_d(k)$, resp. $\eta_d(k)$, the minimum, resp. the maximum , cardinality of minimal transversals for the full subdivisions of $\mathcal{H}(X)$ for all sets X of k points in general position in $\mathbb{R}^d$.

Proposition 1

$$\tau_d(k) = \sum_{i=0}^{d-1} \binom{k-2}{i}, \quad \eta_d(k) = \sum_{i=0}^{d} \binom{k-2}{i}.$$

It is known that $|\mathcal{H}(X)| = \sum_{i=0}^{d} \binom{k-1}{i}$ holds for all sets X of k points in general position in $\mathbb{R}^d$. Let us denote this number by $\phi_d(k)$. We immediately obtain the following equation:

Proposition 2

$$\tau_d(k) + \eta_d(k) = \phi_d(k).$$

Proposition 3

Let X be a set of $k(\geq 2)$ points, which need not be in general position, in $\mathbb{R}^d$. The cardinality of minimal transversals for the full subdivisions of $\mathcal{H}(X)$ is at most $\eta_d(k)$.

A Colorful Kirchberger-type Theorem

Theorem

Let X be a finite subset of $\mathbb{R}^d$, and suppose that each point in X is painted with one of k colors. If every $(d+1)\cdot\eta_d(k)+k$ or fewer points in X can be partitioned along the colors by hyperplanes, then all the points in X can be partitioned along the colors by hyperplanes.

We remark that Arocha et al. [1] and Pór [3] studied other Kirchberger-type theorems, which are different from ours. It is remained as a future work to improve the number $(d+1)\cdot\eta_d(k)+k$ in our theorem.

References

[1] J.L. Arocha, I. Bárány, J. Bracho, R. Fabila, and L. Montejano, "Very colorful theorems," Disc. Comput. Geom., vol.42, pp.142–154, 2009.

[2] P. Kirchberger, "Über tschebyscheffsche annäherungsmethoden," Math. Ann., vol.57, pp.509–540, 1903.

[3] A. Pór, Diploma Thesis, Eötvös University, Budapest, 1998.

[1] We say that a partition P of X can be *realized by a hyperplane* if $P=\{X\}$ or there is a hyperplane h such that $P=\{X\cap h^+, X\cap h^-\}$, where h^+ and h^- are the two open halfspaces associated with h.